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Supersymmetry in Field Theory

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Abstract

Supersymmetric theories are reviewed in the context of field theories. The gauge hierarchy problem in attempting the unification of all fundamental interactions is the strongest motivation of modern development of supersymmetry. Starting from the general notion of supersymmetry as a symmetry between bosons and fermions, we explain how the supersymmetry becomes a part of the space-time symmetry if we wish to maintain the relativistic invariance. The precise idea of supersymmetry is then introduced and the supersymmetric field theories are formulated. There has been a significant breakthrough in the study of nonperturbative effects in supersymmetric field theories using the holomorphy and symmetry arguments. Some of these ideas and results are briefly reviewed.

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1 Motivations for Supersymmetry

1.1 Gauge Hierarchy

1. Standard model

Many efforts have been devoted to study the fundamental constituents of matter and the fundamental interactions between them. At present, the experimental efforts have reached the energy scales of several 1000GeV in collisions between protons and/or antiprotons, and that of a few 100GeV in collisions between electrons and positrons.

It has been found that all the available experimental data up to these energies can be more or less adequately described by the so-called standard model. In the standard model, the fundamental constituents of matter are quarks and leptons and the three known fundamental forces in nature, strong, weak, and electromagnetic interactions are described by a gauge field theory with the $SU(3) \times SU(2) \times U(1)$ gauge group. The standard model succeeded to describe the three fundamental interactions by a common unifying idea called the gauge principle and gave many successful predictions. The most striking confirmation of standard model is the discovery of the weak bosons, W and Z with the mass of the order of $M_W \approx 100\text{GeV}$. However, there are three different gauge coupling constants for each of these gauge groups $SU(3)$, $SU(2)$, and $U(1)$. In that sense, the three different strengths of the three fundamental interactions are parametrized nicely, but are not quite unified. Moreover, the standard model has many input parameters that can only be determined from the experimental measurements. There are also other conceptually unsatisfactory points as well. For instance, the electric charge is found to be quantized in nature, but this phenomenon is just an accident in the standard model.

2. Grand Unified Theories

Because of quantum effects, the effective gauge coupling constants change logarithmically as a function of energy scale. Then there is a possibility that the different gauge couplings for the three fundamental interactions can become the same strength at very high energies M_G . This means that the three gauge interactions can be truly unified into a single gauge group if we choose an appropriate simple gauge group. This idea was proposed by Georgi and Glashow [1], and these models are called the grand unified theories. The grand unified theories achieved at least two good points:

- Because of simple gauge group, the electromagnetic charge is now quantized.
- Two coupling can meet at some point provided they are in the right direction. Since the grand unified theory unifies all three couplings at high energies, it gives one constraint for three couplings. Taking the two measurements of coupling constants at low energies as inputs, one can then predict the third coupling. With the simplest possibility for the unifying gauge group, this prediction was found to be not very far from the

experimental data. On the other hand, the unification energy M_G is now very large compared to the mass scale M_W of the weak boson in the standard model [2]

$$\frac{M_W^2}{M_G^2} \approx \left(\frac{10^2}{10^{16}} \right)^2 \approx 10^{-28} \quad (1.1)$$

3. Gravity

Even if one does not accept the grand unified theories, one is sure to accept the existence of the fourth fundamental force, the gravitational interactions. The mass scale of the gravitational interactions is given by the Planck mass M_{Pl}

$$\frac{M_W^2}{M_{Pl}^2} \approx \left(\frac{10^2}{10^{19}} \right)^2 \approx 10^{-34} \quad (1.2)$$

Now we have a problem of how to explain these extremely small ratios between the mass squared M_W^2 to the fundamental mass squared M_G^2 or M_{Pl}^2 in eq.(1.1) or eq.(1.2). This problem is called the **gauge hierarchy problem**.

1.2 Higgs Scalar

Precisely speaking, when we say **explain** some phenomenon, we mean that it should be given a symmetry reason. This principle is called the naturalness hypothesis [3], [4]. More precisely, the system should acquire higher symmetry as we let the small parameter going to zero. The examples of the enhanced symmetry corresponding to the small mass parameter are

$$\begin{aligned} m_{J=1/2} \rightarrow 0 & \Leftrightarrow \text{Chiral symmetry} \\ m_{J=1} \rightarrow 0 & \Leftrightarrow \text{Local gauge symmetry} \end{aligned} \quad (1.3)$$

The mass scale M_W of weak bosons originates from the vacuum expectation value v of the Higgs scalar field. The scale of v in turn comes from the (negative) mass squared of the Higgs scalar φ . Therefore we need to give symmetry reasons for the extremely small Higgs scalar mass to explain the gauge hierarchy problem.

Classically the vanishing mass for scalar field does lead to an enhanced symmetry called scale invariance. However, it is well known that the scale invariance cannot be maintained quantum mechanically.

Up to now three types of possible solutions have been proposed to explain the gauge hierarchy problem.

1. Technicolor model

We can postulate that there is no elementary Higgs scalar at all. The Higgs scalar in the standard model has to be provided as a composite field at low energies. This option requires nonperturbative physics already at energies of the order of $\text{TeV} = 10^3 \text{ GeV}$. It has been rather difficult to construct realistic models that pass all the test at low energies specially the absence of flavor-changing neutral current. Models with composite Higgs scalar are called Technicolor models [5].

2. Supersymmetry

Another option is to postulate a symmetry between Higgs scalar and a spinor field. Then we can postulate chiral symmetry for the spinor field to make it massless. The Higgs scalar also becomes massless because of the symmetry between the scalar and the spinor. This symmetry between scalar and spinor is called supersymmetry [6]. Supersymmetry as a possible solution of gauge hierarchy problem was proposed concretely in the context of supersymmetric grand unified theories [7] [8] [9] [10], although the use of supersymmetry has been advocated for electroweak interactions earlier [11]. Contrary to the Technicolor models, we can construct supersymmetric models that can be treated perturbatively up to extremely high energies along the spirit of the grand unified theories [12], [13].

Experimental progress for the precise measurements of coupling constants enabled one to test the unification hypothesis precisely. More than 10 years after the initial proposal of supersymmetric grand unified theories, the experimental data from LEP nicely confirmed that the nonsupersymmetric model does not give unification at a single point, and the supersymmetric model gives an excellent agreement [14].

3. Large extra dimensions

The most recent proposal was to note that the gravitational interactions are not tested at short distances below mm. Therefore one can consider the possibility of the fundamental scale of gravitational interactions of 1000GeV . The observed smallness of the gravitational interaction in our world is explained by imagining the extra dimensions compactified at the radius of order mm or less [15]. The supersymmetry is not needed logically in this case, although it is often used to construct concrete models.

1.3 Symmetry Relating Different Statistics and Spin

1.3.1 Symmetry Relating Different Statistics

Supersymmetry can be defined as a symmetry relating bosons and fermions. Namely particles with different statistics are related by the supersymmetry.

There is no significant constraint in formulating such a supersymmetry in nonrelativistic quantum theories. In fact the supersymmetry has been useful in several areas of nonrelativistic quantum theory such as condensed matter physics and nuclear physics. Let us mention two interesting applications:

1. Solid State Physics

If one considers a spin system in random magnetic fields, the randomness of the magnetic field tends to disorder the spin system. It has been found that the critical behavior of the spin system in random magnetic fields in d dimensions is the same as that of the spin system without the random magnetic fields in $d - 2$ dimensions. This phenomenon is sometimes called dimensional reduction. Parisi and Sourlas gave a beautiful explanation of this phenomenon by uncovering the underlying supersymmetry of the spin system in the random magnetic fields [16].

2. Nuclear Physics

In certain complex nuclei, it is quite useful to use supersymmetry among quasi particle excitations to classify various nuclear energy levels.

1.3.2 Symmetry Relating Different Spins

We are mainly interested in supersymmetry as a fundamental symmetry principle. We have two other fundamental principles in modern physics: quantum theory and relativity. In nature, all bosons have integer spin and all fermions have half-odd integer spin. This fact can be explained if we employ relativistic quantum field theory. Therefore supersymmetry inevitably becomes a **symmetry between particles with different spin** if we want to maintain relativistic invariance. Since the spin is a quantum number associated with the rotation, we need to formulate supersymmetry as a symmetry that is nontrivially combined with the space-time symmetry such as rotations, translations, and Lorentz transformations.

It has been a notoriously difficult problem to formulate a nontrivial symmetry that relates particles with different spins. This point can be most neatly summarized by the so-called “No-go Theorem” by Coleman and Mandula [17] They assumed Lorentz invariance, analyticity of scattering amplitudes (corresponding to the causality), nontrivial S-matrix, and other technical assumptions. They found that Poincaré group can only appear as a direct product group with other symmetry. Namely no nontrivial symmetry is possible between particles of different spins. In this No-go theorem, they have actually assumed that all the symmetry relations are expressed in terms of commutation relations.

Much later, it has been found that nontrivial symmetry is possible if one uses anticommutation relations among symmetry generators instead of the ordinary commutation relations. With the same assumptions as those of Coleman and Mandula except the introduction of the anticommutation relation, Haag, Lopuszanski, and Sohnius were able to obtain the most general symmetry [18]. They have found that the supersymmetry as we know now is the only possible symmetry that involves space-time symmetry nontrivially. We will describe this supersymmetry in subsequent sections.

2 Basic Concepts in Supersymmetric Field Theory

2.1 Superfield and Supertransformation

To formulate symmetry such as rotation, it is most convenient to introduce a coordinate system to distinguish different directions in space. Similarly, to formulate the supersymmetry, it is useful to introduce a coordinate θ to distinguish bosons and fermions. It has to be an anticommuting spinor, since it relates bosons and fermions. Our conventions for spinors are summarized in Appendix.A. Anticommuting number is called Grassmann number. Combined with the space-time coordinates x^m , we have x^m, θ as coordinates in superspace.

A function $\Phi(x, \theta)$ of x^m, θ is called superfield. Because of anticommuting property, the superfield can be expanded in terms of Grassmann number to obtain the finite number of ordinary fields. In the case of four component Majorana spinor θ , the superfield contains 16 component of ordinary fields. Half of them are bosons and half of them are fermions.

$$\begin{aligned} \Phi(x, \theta) = & C(x) + \bar{\theta}\psi(x) - \frac{1}{2}\bar{\theta}\theta N(x) - \frac{i}{2}\bar{\theta}\gamma_5\theta M(x) \\ & - \frac{1}{2}\bar{\theta}\gamma^m\gamma_5\theta v_m(x) + i\bar{\theta}\theta\gamma_5\lambda(x) + \frac{1}{4}(\bar{\theta}\theta)^2 D(x) \end{aligned} \quad (2.1)$$

Let us consider as a simplest transformation in the superspace an (infinitesimal) translation by ϵ in the Grassmann number θ . To make it a nontrivial space-time symmetry, we shift also the space-time coordinate as follows,

$$\delta\theta = \epsilon, \quad \delta x^m = -i\bar{\epsilon}\gamma^m\theta \quad (2.2)$$

This form is the simplest possibility that is Lorentz covariant and is linear in ϵ . This transformation is called the supertransformation. With this transformation, the superfield is transformed

as

$$\begin{aligned}\delta\Phi(x, \theta) &= \bar{\epsilon} \left(\frac{\partial}{\partial\theta} - i\gamma^m\theta \frac{\partial}{\partial x^m} \right) \Phi(x, \theta) = - \left(\frac{\partial}{\partial\theta} - i\bar{\theta}\gamma^m \frac{\partial}{\partial x^m} \right) \epsilon \Phi(x, \theta) \\ &\equiv [\Phi(x, \theta), \bar{\epsilon}Q] = [\Phi(x, \theta), \bar{Q}\epsilon]\end{aligned}\quad (2.3)$$

The first line is represented by a differential operator in terms of the Grassmann number acting on superfield, whereas the second line is expressed as a commutator between the quantized superfields and the supercharge Q which is the unitary operator for the supersymmetry transformation. It is useful to note that the basic definition of the supertransformation dictates that the Grassmann number $\theta, \bar{\theta}$ have dimension of the square-root of the coordinate x^m . Useful formulas for derivatives of Grassmann numbers are summarized in the Appendix.B.

To find the algebra satisfied by the supercharges, we make two successive supertransformations in eq.(2.2), and make the difference between the results of transformations in different order

$$\begin{aligned}[\Phi, [\bar{\epsilon}_1 Q, \bar{Q}\epsilon_2]] &= [\Phi, [\bar{\epsilon}_1 Q, \bar{\epsilon}_2 Q]] = [[\Phi, \bar{\epsilon}_1 Q], \bar{\epsilon}_2 Q] - [[\Phi, \bar{\epsilon}_2 Q], \bar{\epsilon}_1 Q] \\ &= (\delta(\epsilon_2))(\delta(\epsilon_1))\Phi - (\delta(\epsilon_1))(\delta(\epsilon_2))\Phi \\ &= \left[\left(-\frac{\partial}{\partial\theta} + i\bar{\theta}\gamma^m\partial_m \right) \epsilon_2, \bar{\epsilon}_1 \left(\frac{\partial}{\partial\theta} - i\gamma^n\theta\partial_n \right) \right] \Phi(x, \theta) \\ &= 2\bar{\epsilon}_1\gamma^m\epsilon_2 (-i\partial_m\Phi(x, \theta)) = 2\bar{\epsilon}_1\gamma^m\epsilon_2 [\Phi(x, \theta), P_m]\end{aligned}\quad (2.4)$$

Thus we find that the anticommutator of the supercharges is given by the space-time translation represented by the four-momentum operator P^m . This property is a direct consequence of the space-time coordinate shift bilinear in Grassmann numbers in eq.(2.2).

Since the chirality projection is useful in formulating supersymmetry, we shall use the two component notation for spinors from now on. The two component notation is summarized in Appendix.A. Then the anticommutators between supercharges are given by

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^m)_{\alpha\dot{\beta}}P_m, \quad \{Q_\alpha, Q_\beta\} = 0, \quad \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \quad (2.5)$$

The translation operator P^m together with the Lorentz transformations J^{mn} form the group of space-time transformations, the Poincaré group. The other commutation relations are found to have intuitive physical meaning. First the supercharges are translation invariant and transform as a spinor under the Lorentz transformations.

$$[Q, P_m] = 0, \quad [Q_\alpha, J^{mn}] = i(\sigma^{mn})_\alpha{}^\beta Q_\beta, \quad [\bar{Q}^{\dot{\alpha}}, J^{mn}] = i(\bar{\sigma}^{mn})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{\dot{\beta}} \quad (2.6)$$

The rest of the algebra forms the ordinary algebra for the Poincaré group.

$$[P_m, P_n] = 0, \quad [P^m, J^{nl}] = -i(\eta^{mn}P^l - \eta^{ml}P^n) \quad (2.7)$$

$$[J^{mr}, J^{nl}] = -i(\eta^{rn} J^{ml} + \eta^{ml} J^{rn} - \eta^{mn} J^{rl} - \eta^{rl} J^{mn}) \quad (2.8)$$

Thus we find, as promised, that the supersymmetry has two characteristic features:

1. It involves the anticommutators. and
2. It is a part of spacetime symmetry.

2.2 Unitary Representation

Supersymmetry requires bosons and fermions to form a multiplet. To find the particle content dictated by the supersymmetry explicitly, we need to study the unitary representation of the supersymmetry algebra.

2.2.1 $N = 1$ Massive case

Since the supersymmetry is a part of the space-time symmetry, we should combine unitary representations of Poincaré group to form the unitary representation of the supersymmetry. To obtain the unitary representation of the Poincaré group, we first diagonalize the four momentum P^m . For the massive case, we can choose the rest frame as the standard frame $P^m = (M, 0, 0, 0)$. The stability group that leaves the standard frame $P^m = (M, 0, 0, 0)$ unchanged is the $SO(3)$ subgroup. The unitary representation of the $SO(3)$ subgroup is labeled by the angular momentum j and its z component m . Now we should combine these representations (P^m, j, m) of the Poincaré group to obtain the unitary representation of the supercharge Q , since Q commutes with the four momenta $[Q, P_m] = 0$. Since the supercharge has spin $1/2$ as shown in eq.(2.6), Q changes j and m by $\pm \frac{1}{2}$. The anticommutators (2.5) between supercharges Q are precisely the same algebra as the fermion creation and annihilation operators, if we rescale by $\sqrt{2M}$.

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, \quad \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2M\delta_{\alpha\dot{\beta}} \quad (2.9)$$

Since there are 2 components of spinor indices, there are 2 kinds of “fermions”. We can regard $\bar{Q}_{\dot{\alpha}}$, $\dot{\alpha} = 1, 2$ as “annihilation operators”, and Q_α , $\alpha = 1, 2$ as “creation operators”. The unitary representations of these operators can be obtained by assuming ground state that is defined as the state annihilated by the “annihilation operators” $\bar{Q}_{\dot{\alpha}}|j\rangle = 0$, $\alpha = 1, 2$. Here the ground state $|j\rangle$ is assumed to be an eigenstate of angular momentum j . Since the multiplication of the same type of supercharges vanish, we obtain only four possible states by applying the “creation operator” Q_α

$$\begin{pmatrix} |j\rangle & Q_1|j\rangle & Q_1Q_2|j\rangle \\ & Q_2|j\rangle & \end{pmatrix} \sim \begin{pmatrix} j - \frac{1}{2} & \\ j & j \\ j + \frac{1}{2} & \end{pmatrix} \quad (2.10)$$

The number of states in the multiplet is given by $4(2j+1)$, $j = 0, \frac{1}{2}, \dots$. Two lowest multiplets of the massive supermultiplet are explicitly shown in the table.

1. $j = 0$ case \Rightarrow Chiral scalar multiplet

spin j	field	degree of freedom
0	two real scalar	2
1/2	a Majorana spinor	2

2. $j = \frac{1}{2}$ case \Rightarrow Vector multiplet

spin j	field	degree of freedom
0	a real scalar	1
1/2	2 Majorana spinor	4
1	a real vector	3

2.2.2 $N = 1$ Massless case

In the case of massless particles, we can choose the standard frame as $P^m = (P, 0, 0, P)$. The stability group that leaves the standard frame $P^m = (P, 0, 0, P)$ unchanged is the Euclid group in two dimensions E_2 : $E_2 = (J^{12}, J^{01} - J^{31}, J^{02} + J^{23})$. It is well-known that the unitary representation of massless particles is labeled by the helicity J^{12} [19]. In the standard frame, the nonvanishing anticommutator between supercharges is given by

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma_0 + \sigma_3)_{\alpha\dot{\beta}} P = 4P \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.11)$$

$$\{Q_1, \bar{Q}_1\} = 4P, \quad \bar{Q}_1 = (Q_1)^* \quad (2.12)$$

Therefore we have only single fermion “creation and annihilation operators”. If we take the state of helicity λ as the ground state $\bar{Q}_1|\lambda\rangle = 0$, we obtain a multiplet consisting of only 2 states whose helicities differ by $1/2$.

$$(|\lambda\rangle, Q_1|\lambda\rangle) \sim \left(|\lambda\rangle, |\lambda - \frac{1}{2}\rangle\right) \quad (2.13)$$

Although the number of states in a multiplet is two, it is often required that the CPT invariance necessitates to combine states with opposite helicity if they are not in the same multiplet. Then the number of states becomes four. Frequently used multiplets are shown in the table.

$$(\lambda, \lambda - \frac{1}{2}, -\lambda + \frac{1}{2}, -\lambda) \quad (2.14)$$

highest helicity	helicities of fields	name of multiplet
$\lambda = \frac{1}{2}$	$(\frac{1}{2}, 0, 0, -\frac{1}{2})$	chiral scalar multiplet
$\lambda = 1$	$(1, \frac{1}{2}, -\frac{1}{2}, -1)$	vector multiplet
$\lambda = 2$	$(2, \frac{3}{2}, -\frac{3}{2}, -2)$	graviton multiplet

2.2.3 Extended Supersymmetry

The most general supersymmetry algebra found by Haag et. al. contains N species of supercharges Q^L [18]. It is called the N -extended supersymmetry. In two component notation, it reads

$$\{Q_\alpha^L, \bar{Q}_{\dot{\beta}M}\} = 2(\sigma^m)_{\alpha\dot{\beta}} P_m \delta_M^L, \quad \{Q_\alpha^L, Q_\beta^M\} = \epsilon_{\alpha\beta} X^{LM}, \quad \{\bar{Q}_{\dot{\alpha}L}, \bar{Q}_{\dot{\beta}M}\} = \epsilon_{\dot{\alpha}\dot{\beta}} X_{LM}^\dagger \quad (2.15)$$

$$[X^{KL}, Q_\alpha^M] = [X^{KL}, \bar{Q}_{\dot{\alpha}M}] = [X^{LM}, X^{KN}] = 0 \quad (2.16)$$

where X are called central charges.

1. Let us first consider the massless case without central charges X . Similarly to eq.(2.11), the N -extended supersymmetry gives Q_1^L , $L = 1, \dots, N$ as fermion “creation operators”, if there are no central charges. Starting from the ground state with the helicity λ , we descend in helicity by half unit in each step by operating Q_1^L .

$$\begin{array}{ccccccc}
|\lambda\rangle \rightarrow & |\lambda - \frac{1}{2}\rangle & \rightarrow & |\lambda - 1\rangle & \cdots & \rightarrow & |\lambda - \frac{N}{2}\rangle \\
1 & & & N & & & \left(\begin{array}{c} N \\ 2 \end{array} \right) \quad \cdots \quad 1
\end{array} \quad (2.17)$$

The number of states is denoted below each helicity states and sums to 2^N . If the multiplet is not CPT self-conjugate, CPT conjugate states should be added. Two points are worth mentioning:

- There are a number of arguments suggesting that consistent formulation of interacting massless fields is limited to spin up to two in four-dimensions. This limits the highest helicity to be less than or equal to 2.

$$|\lambda| \leq 2, \quad |\lambda - \frac{N}{2}| \leq 2 \quad (2.18)$$

Therefore the highest possible supersymmetry is $N = 8$ which gives $4 \times 8 = 32$ supercharges. The $N = 8$ supersymmetry in four-dimension is maximal, and it automatically contains graviton ($\lambda = \pm 2$). Therefore the interacting $N = 8$ supersymmetric theory is nothing but the $N = 8$ extended supergravity.

- If one wants a renormalizable theory, highest helicity should be one or less. This limits N to be less than or equal to $N = 4$: $J \leq 1 \Rightarrow N \leq 4$. The maximal case gives the maximally supersymmetric gauge theory: $N = 4$ supersymmetric Yang-Mills theory.
2. Massive N -extended supersymmetry case without central charge X allows $2N$ supercharges Q_1^L, \dots, Q_N^L , $L = 1, \dots, N$ as fermion “creation operators”. We thus obtain the number of states in a multiplet to be $2^{2N}(2j+1)$, $j = 0, \frac{1}{2}, \dots$.
3. BPS states:

If we have massive N -extended supersymmetry case with central charge X , we can have interesting situation called the BPS states where only a part of supersymmetry is maintained giving the smaller number of states in a single multiplet. Such a multiplet is sometimes called a short representation.

Let us illustrate by an example in the $N = 2$ case that has $SU(2)$ as an internal symmetry. Since the central charge has to be proportional to the invariant tensor of the internal symmetry $SU(2)$, we parametrize

$$X^{LM} = 2Z\epsilon^{LM} \quad (2.19)$$

Let us take the rest frame $P^m = (M, 0, 0, 0)$. The $N = 2$ supersymmetry algebra becomes

$$\{Q_\alpha^L, \bar{Q}_{\dot{\beta}M}\} = 2M\delta_{\alpha\dot{\beta}}\delta_M^L, \quad \{Q_\alpha^L, Q_\beta^M\} = 2Z\epsilon_{\alpha\beta}\epsilon^{LM}, \quad \{\bar{Q}_{\dot{\alpha}L}, \bar{Q}_{\dot{\beta}M}\} = 2Z^*\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{LM} \quad (2.20)$$

Then we have to consider both chirality of supercharges together. Since the anticommutator matrix must be positive definite matrix, we obtain an inequality

$$M \geq |Z| \quad (2.21)$$

This bound is called the BPS (Bogomolnyi-Prasad-Sommerfield) bound [20].

If $M = |Z|$, there are zero eigenvalues for the matrix. This implies that a linear combination of Q 's annihilates all the states, and cannot be used to create physical states. Therefore we obtain smaller number of particle states to represent the supersymmetry algebra. For example, if we have $Z = M$, we find convenient linear combinations of supercharges as

$$Q^{(1)} \equiv Q_1^{L=1} + \bar{Q}_{2L=2}, \quad Q^{(2)} \equiv Q_2^{L=1} - \bar{Q}_{1L=2}, \quad (2.22)$$

These satisfy

$$\{Q^{(i)}, Q^{(j)\dagger}\} = 4M\delta^{ij} \quad (2.23)$$

and all other anticommutators vanish.

This is algebraically the same as the case of the massive $N = 1$ supersymmetry. Therefore the number of states is reduced by $1/4$: $2^4(2j+1) \rightarrow 2^2(2j+1)$.

This phenomenon occurs when the determinant of the anticommutators of supercharges vanishes. The resulting multiplet contains a smaller number of physical states and is called the BPS saturated states [21].

The physical origin of the central charge is often given by various nonperturbative objects such as monopoles, dyons, domain walls, in general some kind of solitons.

2.3 Field Theory Realization

2.3.1 Irreducible Representation

The smallest unitary representation of the $N = 1$ supersymmetry in four space-time dimensions requires two real spin 0 particles and two spin $1/2$ particles. On the other hand, the general superfield $\Phi(x, \theta, \bar{\theta})$ has 8 boson fields and 8 fermion fields, as we have seen in eq.(2.1).

To obtain smaller number of components than the general superfield, we should find a constraint that is compatible with the supersymmetry transformation to realize the supersymmetry in a smaller space. This is a key ingredient to construct supersymmetric field theories.

We note that the general spinors θ_α in four space-time dimensions has four components, whereas the chirally projected spinors $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$ have only two components. Therefore if we can construct a superfield that depends only on the chirally projected spinors, we should be able to reduce the number of component fields to half of those of the general superfield. Therefore we are tempted to use the constraint that the superfield be independent of the Grassmann number with one of the chirality

$$\frac{\partial}{\partial \theta^{\dot{\alpha}}} \Phi(x, \theta, \bar{\theta}) = 0 \quad (2.24)$$

Unfortunately even if this constraint is imposed, it is not satisfied after the supersymmetry transformation.

$$\left\{ \frac{\partial}{\partial \theta^{\dot{\alpha}}}, Q_\beta \right\} \neq 0 \quad (2.25)$$

Therefore this constraint is not consistent with supersymmetry. We can modify the derivative with respect to the Grassmann number by an additional term. We define the following covariant

derivatives

$$\bar{D}_{\dot{\alpha}} \equiv -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i\theta^{\alpha}\sigma_{\alpha\dot{\alpha}}^m \partial_m \quad (2.26)$$

$$D_{\dot{\alpha}} \equiv \frac{\partial}{\partial \theta^{\dot{\alpha}}} + i\sigma_{\alpha\dot{\alpha}}^m \bar{\theta}^{\dot{\alpha}} \partial_m \quad (2.27)$$

These covariant derivatives anticommute with the supersymmetry transformation

$$\{D_{\alpha}, Q_{\beta}\} = \{D_{\alpha}, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, Q_{\beta}\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \quad (2.28)$$

Therefore they can be used to constrain the superfield to reduce the number of component fields by half.

$D_{\alpha}, \bar{D}_{\dot{\alpha}}$ satisfy the same algebra as $Q_{\alpha}, \bar{Q}_{\dot{\alpha}}$.

$$\{D_{\alpha}, \bar{D}_{\dot{\alpha}}\} = -2i\sigma_{\alpha\dot{\alpha}}^m \partial_m, \quad \{D_{\alpha}, \bar{D}_{\dot{\beta}}\} = \{D_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0 \quad (2.29)$$

2.3.2 Chiral Scalar Field

By using the covariant derivatives, we can now define the superfield which has half as many components as the general superfields in eq.(2.1). Since the supercharge anticommute with the covariant derivative as shown in eq.(2.28), these chiral scalar fields can be used as a representation space of supersymmetry.

The (negative) chiral scalar superfield is defined by

$$\bar{D}_{\dot{\alpha}}\Phi(x, \theta, \bar{\theta}) = 0 \quad (2.30)$$

We can easily see that the following combination of variables satisfies this constraint

$$y^m \equiv x^m + i\theta\sigma^m\bar{\theta}, \quad \bar{D}_{\dot{\alpha}}y^m = 0 \quad (2.31)$$

Therefore the general solution of the constraint is simply that the superfield depends on the $\bar{\theta}$ only through the combination $y^m \equiv x^m + i\theta\sigma^m\bar{\theta}$.

$$\Phi(y, \theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \quad (2.32)$$

The supertransformation of the chiral scalar superfield is given by means of the derivative operator defined in eq.(2.3). In the two component notation, we obtain

$$\delta_{\xi}\Phi(y, \theta) = \left[\xi^{\alpha} \left(\frac{\partial}{\partial \theta^{\alpha}} - i\sigma_{\alpha\dot{\alpha}}^m \bar{\theta}^{\dot{\alpha}} \frac{\partial}{\partial x^m} \right) + \left(-\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + i\theta^{\alpha}\sigma_{\alpha\dot{\alpha}}^m \frac{\partial}{\partial x^m} \right) \bar{\xi}^{\dot{\alpha}} \right] \Phi(x, \theta) \quad (2.33)$$

In terms of the component fields, we find

$$\delta_\xi A = \sqrt{2}\xi\psi \quad (2.34)$$

$$\delta_\xi \psi = i\sqrt{2}\sigma^m \bar{\xi} \partial_m A + \sqrt{2}F \quad (2.35)$$

$$\delta_\xi F = i\sqrt{2}\bar{\xi} \bar{\sigma}^m \partial_m \psi \quad (2.36)$$

It is important to note that the last component F is transformed into a derivative of the lower component. The supertransformation should increase the mass dimension by $M^{\frac{1}{2}}$. However, the last component has the highest mass dimension and there is no component fields available except to consider the derivative of the lower component fields. This point is always true for the last component of the superfields. Hence the last component of the general superfield also transforms into a total derivative of lower component fields.

It is important to realize that the chiral scalar field is complex. Therefore the scalar component A is a complex scalar field, and the fermionic component ψ is a complex Weyl spinor. Let us count the number of the degrees of freedom of component fields. If we do not use the equation of motion, there are two real scalar components from A and two real scalar components from F , and four real fermionic components from ψ . We call this situation off-shell. Later we will construct the Lagrangian for this chiral scalar field. There we will find that ψ obeys the Dirac equation which reduces the on-shell degrees of freedom to half. Namely we have only a left-handed fermion and its anti-particle. As we noted previously, the mass dimension of the Grassmann number $\theta, \bar{\theta}$ is $M^{-\frac{1}{2}}$. Therefore the mass dimension of the field F is M^2 if we take the mass dimension of the scalar component A to be M^1 as ordinarily required for the scalar field. As we will find when constructing the Lagrangian, this implies that the F cannot have ordinary kinetic term with two derivatives and is an auxiliary field that can be expressed in terms of other fields. We summarize the counting of the number of degrees of freedom in the table.

fields	real or complex spin	off-shell real d.o.f.	on-shell real d.o.f.
A	complex scalar	2	2
ψ	complex 2-comp. spinor	4	2
F	complex aux. scalar	2	0

Similarly, we can define the positive chiral scalar superfield by

$$D_\alpha \Phi(x, \theta, \bar{\theta}) = 0 \quad (2.37)$$

The general solution of the constraint is given by

$$\Phi(y^*, \bar{\theta}) = A^*(y^*) + \sqrt{2}\bar{\theta}\bar{\psi}(y^*) + \bar{\theta}\bar{\theta}F^*(y^*) \quad (2.38)$$

Clearly the product of chiral scalar superfields is still a chiral scalar superfield as long as the chirality is the same. On the other hand, the product of positive chiral and negative chiral scalar fields is a general superfield (without a definite chirality).

The complex conjugation changes the chirality, since the complex variable y^m is changed into $(y^m)^*$ and the chirality of spinor is also changed by the complex conjugation $(\theta)^* = \bar{\theta}$

$$(\Phi(y, \theta))^* = A^*(y^*) + \sqrt{2}\bar{\theta}\bar{\psi}(y^*) + \bar{\theta}\bar{\theta}F^*(y^*) \quad (2.39)$$

2.3.3 Lagrangian Field Theory with Chiral Scalar Fields

As we noted in sect.2.3.2, the last components of superfields always transform into a total derivative. There are two possibilities for the superfields: chiral scalar superfield and general superfield. Therefore we have two candidates for the Lagrangian invariant under supersymmetry transformation up to a total divergence:

1. D -term of general superfield Φ in eq.(2.1)

$$[\Phi(\theta, \bar{\theta})]_D = \frac{1}{4}D^2\bar{D}^2\Phi(\theta, \bar{\theta}) \quad (2.40)$$

Since the product of chiral scalar superfield with opposite chirality is a general superfield, we can take the D term of the product.

2. F_{\pm} -term of chiral scalar superfield $\Phi(\theta), \bar{\Phi}(\bar{\theta})$

$$[\Phi]_F = \frac{1}{2}D^2\Phi, \quad [\bar{\Phi}]_{\bar{F}} = \frac{1}{2}(\bar{D})^2\bar{\Phi} \quad (2.41)$$

Let us consider Lagrangian field theory consisting of chiral scalar fields. Since the supertransformation does not leave any product of chiral scalar fields invariant, we have to be satisfied with the invariance up to total divergence.

It is quite useful to examine the dimensions of various fields. To give the canonical dimension to the scalar component $[A] = M^1$, we usually assume the dimension of the chiral scalar fields to be M^1 .

$$[\Phi(\theta)] = [\Phi(\bar{\theta})] = M^1 \quad (2.42)$$

Since the mass dimensions of the Grassmann number is half of that of the coordinates,

$$[\theta] = [\bar{\theta}] = L^{\frac{1}{2}} = M^{-\frac{1}{2}}, \quad (2.43)$$

we obtain that the covariant derivative has the mass dimensions as $M^{\frac{1}{2}}$

$$[D] = [\bar{D}] = M^{\frac{1}{2}} \quad (2.44)$$

A renormalizable Lagrangian in four space-time dimensions requires that the Lagrangian should consist of operators with dimension ≤ 4 . We can list possible terms as follows.

1. D-type:

$$\bar{D}^2 D^2 \Phi \bar{\Phi} \quad (2.45)$$

Since the mass dimension of the product of covariant derivatives is $[\bar{D}^2 D^2] = M^2$, we see that there are no terms of this class.

2. F-type:

$$D^2(a\Phi_1 + b\Phi_1\Phi_2 + c\Phi_1\Phi_2\Phi_3) = D^2 P(\Phi) \quad (2.46)$$

Since D^2 has dimension M^1 , up to third order polynomials of chiral scalar superfields of one chirality are renormalizable. To maintain the hermiticity of the Lagrangian, we need to add hermitian conjugate terms which consist of the chiral scalar fields of opposite chirality with conjugate coefficients. The polynomial of chiral scalar superfield of the same chirality is called superpotential P .

Now let us illustrate the above consideration with a simple example: general Lagrangian with a single chiral scalar field

$$L = L_{\text{kin}} + L_{\text{int}}. \quad (2.47)$$

$$\begin{aligned} L_{\text{kin}} &= \frac{1}{4} D^2 \bar{D}^2 \Phi^* \Phi \\ &= \frac{1}{4} \partial^2 A^* A - \frac{1}{2} \partial_\nu A^* \partial^\nu A + \frac{1}{4} A^* \partial^2 A \\ &\quad + F^* F + \frac{1}{2} i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi - \frac{1}{2} i \partial_\mu \bar{\psi} \bar{\sigma}^\mu \psi \\ &= -\partial_\nu A^* \partial^\nu A - i \partial_\mu \bar{\psi} \bar{\sigma}^\mu \psi + F^* F + \text{total derivatives} \end{aligned} \quad (2.48)$$

$$\begin{aligned}
L_{\text{int.}} &= \frac{1}{4}D^2 \left(\frac{1}{3}f\Phi^3 + \frac{m}{2}\Phi^2 + \text{h.c.} + s\Phi \right) \\
&= f \left(FA^2 - \psi\psi A \right) + m \left(FA - \frac{1}{2}\psi\psi \right) + sF + \text{h.c.}
\end{aligned} \tag{2.49}$$

The Euler-Lagrange equation for F is given by

$$F^* + fA^2 + mA + s = 0 \tag{2.50}$$

By solving this equation, we can eliminate the auxiliary field F from the Lagrangian L

$$\begin{aligned}
L \rightarrow & -\partial_\nu A^* \partial^\nu A + \frac{1}{2}\bar{\psi}i\bar{\sigma}^\mu\partial_\mu\psi - \frac{m}{2}\bar{\psi}\psi \\
& - (f\psi\psi A^* + \text{h.c.}) - |fA^2 + mA + s|^2
\end{aligned} \tag{2.51}$$

Let us suppose temporarily that the vacuum expectation value of the scalar field A vanishes. Then the parameter m gives the mass of a Majorana spinor ψ and a complex scalar A . The parameter f gives the Yukawa coupling and the scalar four point coupling $|A^2|^2$ in the potential.

2.3.4 Supersymmetric gauge theory

Ordinary local gauge transformation for the matter field $\psi(x)$ in the representation corresponding to a matrix T^a is given by

$$\psi(x) \rightarrow e^{-i\Lambda^a(x)T^a}\psi(x) \tag{2.52}$$

The matter field should be extended to a chiral scalar superfield $\Phi(x, \theta)$ in the supersymmetric theory. In order to maintain chirality of the superfield, we need to extend the gauge parameter function $\Lambda(x)$ to be a chiral scalar superfield $\Lambda(x, \theta)$.

$$\Phi(x, \theta) \rightarrow \exp(-i\Lambda^a(x, \theta)T^a)\Phi(x, \theta) \tag{2.53}$$

Since the chiral scalar superfield contains a complex scalar field, supersymmetrized local gauge transformation actually contains scale transformations.

The kinetic term of the matter fields should be made gauge invariant by introducing the gauge field. In supersymmetric field theory, the kinetic term of the chiral scalar fields consists of product of chiral scalar field with opposite chirality $\Phi^*\Phi$ as in eq.(2.48). Therefore we need to introduce a general superfield as in eq.(2.1) instead of chiral scalar superfield. We see immediately that the general superfield contains vector field as a component. For this reason, the general superfield is

sometimes called the vector superfield. The vector superfield V can be expanded in terms of $\theta, \bar{\theta}$ to obtain component fields

$$\begin{aligned} V(x, \theta) \equiv & C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) \\ & + \frac{i}{2}\theta\theta(M + iN) - \frac{i}{2}\bar{\theta}\bar{\theta}(M - iN) - \theta\sigma^m\bar{\theta}v_m(x) + i\theta\theta\bar{\theta}(\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^m\partial_m\chi(x)) \\ & - i\bar{\theta}\bar{\theta}\theta(\lambda(x) + \frac{i}{2}\sigma^m\partial_m\bar{\chi}(x)) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left(D(x) + \frac{1}{2}\partial^2 C(x)\right) \end{aligned} \quad (2.54)$$

With this vector superfield, the supersymmetric version of the gauge transformations is given by

$$e^{2gV} \rightarrow e^{-i\Lambda^\dagger} e^{2gV} e^{i\Lambda} \quad (2.55)$$

Here the general superfield $V \equiv V^a T^a$ belongs to the adjoint representation of the gauge group and g is the gauge coupling constant. It is dimensionless and real.

$$V^{a*} = V^a \quad (2.56)$$

With this gauge transformation, the kinetic term for the chiral scalar superfield becomes gauge invariant.

$$\text{tr}(\bar{\Phi} e^{2gV} \Phi) \rightarrow \text{tr}(\bar{\Phi} e^{2gV} \Phi) \quad (2.57)$$

In order to examine the gauge transformation of the vector supermultiplet, it is simplest if we consider the U(1) case

$$V \rightarrow V + \frac{i}{2g}(\Lambda - \Lambda^*) \quad (2.58)$$

$$\Lambda = A + \sqrt{2}\theta\psi + \theta\theta F \quad (2.59)$$

v^m is an ordinary gauge field with $\text{Re}A$ as the ordinary (real) gauge transformation parameter

$$v^m \rightarrow v^m + \frac{1}{2g}\partial^m(A + A^*) \quad (2.60)$$

λ, D are gauge invariant.

$$\begin{aligned} \lambda &\rightarrow \lambda \\ D &\rightarrow D \end{aligned} \quad (2.61)$$

C, χ, M, N can be gauged away by $\text{Im}A, \psi, F$ in supersymmetric gauge parameter superfield Λ

$$\begin{aligned} C &\rightarrow C + \frac{i}{2g}(A - A^*) \\ \chi &\rightarrow \chi + \sqrt{2}\frac{1}{2g}\psi \\ M + iN &\rightarrow M + iN + \frac{1}{2g}F \end{aligned} \tag{2.62}$$

By exploiting the supersymmetric version of the gauge transformation, we can go to the Wess-Zumino gauge that is most popular to unravel the physical particle content of the model.

$$\begin{aligned} V_{WZ} &= -\theta\sigma^m\bar{\theta}v_m(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) \\ &\quad - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \end{aligned} \tag{2.63}$$

Since we have used the gauge transformation to go to the Wess-Zumino gauge, the Wess-Zumino gauge is not manifestly supersymmetric. In this gauge, supersymmetry is no longer manifest, but the invariance under the ordinary gauge transformation remains. The particle content can be most easily seen in the Wess-Zumino gauge.

To form a Lagrangian, we need to build the gauge field strength as a gauge covariant building block. Among component fields of the vector superfield V , the gaugino field $\lambda^a(x)$ is the gauge covariant field with lowest dimension. We can obtain this component by applying the covariant derivative D once and \bar{D} twice.

$$W_\alpha \equiv \frac{-1}{8g}(\bar{D}\bar{D})\left(e^{-2gV^aT^a}D_\alpha e^{2gV^aT^a}\right) = -i\lambda_\alpha + \dots \tag{2.64}$$

Since we have differentiated twice in $\bar{\theta}$, W_α is a negative chiral superfield and gauge covariant

$$\bar{D}_{\dot{\beta}}W_\alpha = 0 \tag{2.65}$$

$$W_\alpha \rightarrow e^{-i\Lambda^aT^a}W_\alpha e^{i\Lambda^aT^a} \tag{2.66}$$

Similarly a positive chiral field strength is given by

$$\bar{W}_{\dot{\alpha}} = \frac{-1}{8g}(DD)(e^{2gV^aT^a}\bar{D}_{\dot{\alpha}}e^{-2gV^aT^a}) \tag{2.67}$$

Since supersymmetric gauge field strength is a chiral superfield, the kinetic term for vector superfield is given by the F term of the square of the supersymmetric field strength

$$L_{\text{gauge}} = \frac{1}{8}D^2(W^\alpha W_\alpha) + \text{h.c.} \tag{2.68}$$

In the Wess-Zumino gauge, the Lagrangian is given in terms of the component fields as

$$L_{\text{gauge}} = i\bar{\lambda}\sigma^m\partial_m\lambda - \frac{1}{4}v_{\mu\nu}^a v^{a\mu\nu} + \frac{1}{2}D^a D^a \quad (2.69)$$

$$v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu + ig[v_\mu, v_\nu] \quad (2.70)$$

$$\nabla_\mu\lambda = \partial_\mu\lambda + ig[v_\mu, \lambda] \quad (2.71)$$

Similarly to the F fields, the last component D^a is an auxiliary field.

2.4 The General $N = 1$ Supersymmetry Lagrangian up to two Derivatives

Since we are interested in effective action, we should not require the action to be renormalizable. Here we will write down the most general $N = 1$ supersymmetry Lagrangian in flat space (without gravity) which has up to two derivatives of fields. We have the following building blocks

1. Field content

Chiral superfield Φ

Vector superfield V

2. Superpotential $P(\Phi)$

Interaction between chiral scalar fields are given by a function called superpotential which depends on the chiral scalar fields of the same chirality only.

3. Kähler potential $K(\Phi^\dagger, \Phi)$

The kinetic term of the chiral scalar superfields is given by the D term of a general superfield that is given by a function of chiral scalar superfields of both chirality. Since the D term is taken, the kinetic term of the action is unchanged by a transformation with a function $f(\Phi)$ and $\bar{f}(\bar{\Phi})$

$$K(\Phi^\dagger, \Phi) \rightarrow K(\Phi^\dagger, \Phi) + f(\Phi) + \bar{f}(\bar{\Phi}) \quad (2.72)$$

This invariance is called Kähler invariance. This function can be regarded as giving a geometry of field space of the chiral scalar superfields. This geometry is called Kähler metric and the function is called Kähler potential. Additional term due to the gauge interaction is denoted as Γ .

4. Gauge kinetic function $H_{ab}(\Phi)$

Since the gauge kinetic term is given by the F term of supersymmetric gauge field strength, it can be multiplied by a function of chiral scalar fields which is called the gauge kinetic function.

5. Fayet-Iliopoulos D-term for $U(1)$ ξ

Since $U(1)$ vector superfield is neutral, the D term of the vector superfield is neutral and transforms into total derivative under the supertransformation. Therefore one can add a D term of the $U(1)$ vector superfield itself $V^{(1)}$ into the Lagrangian.

We shall denote the F -type term as $|_{\theta\theta}$ or $|\bar{\theta}\bar{\theta}$ and D -type term as $|\theta\theta\bar{\theta}\bar{\theta}$

$$\begin{aligned}\mathcal{L} = & \left[K(\Phi^\dagger, \Phi) + \Gamma(\Phi^\dagger, \Phi; V) \right] \Big|_{\theta\theta\bar{\theta}\bar{\theta}} \\ & + \left(\frac{1}{4} H_{ab}(\Phi) W^{a\alpha} W_\alpha^b \Big|_{\theta\theta} + h.c. \right) + 2\xi V^{(1)} \Big|_{\theta\theta\bar{\theta}\bar{\theta}} \\ & + (P(\Phi) \Big|_{\theta\theta} + h.c.)\end{aligned}\tag{2.73}$$

The minimal forms of the Kähler potential and gauge kinetic function are given by

$$K + \Gamma = \Phi^\dagger e^{2V} \Phi,\tag{2.74}$$

$$H_{ab} = \frac{1}{g^2} \delta_{ab}\tag{2.75}$$

On the other hand, an interesting example of the nonminimal gauge kinetic function is given by

$$H_{ab}(S) = \frac{1}{g^2} \delta_{ab} + S \delta_{ab} + \cdots\tag{2.76}$$

where S is a chiral scalar superfield which is a singlet of the gauge group. The mass dimension of the chiral scalar superfield and the superpotential P is

$$[\Phi] = M, \quad [P(\Phi)] = M^3\tag{2.77}$$

If renormalizability is required, the superpotential $P(\Phi)$ should be cubic or less in Φ .

The equation of motion for the auxiliary field F^{j*} is given by

$$g_{ij*} F^i - \frac{1}{2} g_{kj*} \Gamma_{ml}^k \chi^m \chi^l + \frac{\partial P^*}{\partial A^{*j}} = 0\tag{2.78}$$

$$g_{ij*} = \frac{\partial^2 K}{\partial A^i \partial A^{*j}} \quad (2.79)$$

$$\Gamma_{ml}^k = g^{kn*} \frac{\partial}{\partial A^l} g_{mn*} = g^{kn*} \frac{\partial^3 K}{\partial A^l \partial A^m \partial A^{*n}} \quad (2.80)$$

The equation of motion for auxiliary field D for minimal kinetic term is given by

$$\frac{1}{g} D^a + \Sigma_k A^{*k} T^a A^k = 0 \quad (2.81)$$

$$\frac{1}{e} D + \Sigma_k A^{*k} Q A^k + \xi = 0 \quad (2.82)$$

2.5 Perturbative Nonrenormalization Theorem

It has been very useful to use superfield perturbation theory to organize the perturbative corrections. The most interesting prediction of the superfield perturbation was the nonrenormalization theorems [22] [23]. Since the interaction among chiral scalar superfield consists of superfields with the same chirality, there is a selection rule based on purely algebraic identities on the chirality structure of possible loop corrections. By performing the algebra of Grassmann numbers, it has been shown that the loop corrections to all orders of perturbation do not give any F -type terms. This implies that not only the divergent terms but also finite terms do not appear in the F -type terms. The loop corrections in quantum effects appear only in the D -type terms. Therefore the following local terms can be generated in quantum effects.

1. Kähler potential $K(\Phi, \bar{\Phi})$

This gives the kinetic term of chiral scalar multiplet

2. Gauge kinetic function $H_{ab}(\Phi)$

This can give the nonminimal kinetic term for vector multiplet. Although the gauge kinetic term is written as a F -type term, the gauge field strength actually involves the covariant derivative of opposite chirality. Therefore it can be generated in loop corrections.

3. Fayet-Iliopoulos D-term for $U(1)$

As a consequence, we obtain the following

1. No quadratic divergences

Typically the mass parameter can get quadratic divergences, but there is no loop corrections at all for parameters appearing in superpotential such as the mass parameters.

2. No quantum corrections to masses and Yukawa couplings

For such parameters in the superpotential, even a finite correction is absent.

3. Only wave function renormalization and gauge coupling renormalization are needed.

They are typically logarithmically divergent.

Let us emphasize that the necessity of the wave function renormalization means that the parameters such as mass, Yukawa coupling constant still run as one changes the scale. Therefore it is still meaningful to consider these parameters as effective coupling constants that depend on the energy scale. It should also be stressed that the above nonrenormalization theorem is obtained by the perturbation theory and is valid to all orders of perturbation. Therefore the nonperturbative effects can violate the nonrenormalization theorem.

Another interesting perturbative result is that the beta function is exactly given by 1-loop in the $N = 2$ supersymmetric gauge theories [24].

2.6 R-symmetry

In supersymmetric theories, one can define a new type of symmetry called the R-symmetry. This is a continuous global symmetry that rotates phases of all the fermions relative to all the bosons. This is most easily achieved by a rotation of Grassmann numbers.

$$\theta \rightarrow e^{-i\epsilon}\theta \quad (2.83)$$

At the same time, one can assign an R-charge for chiral scalar superfield Φ : R_Φ .

$$\Phi(\theta) \rightarrow e^{i\epsilon R_\Phi} \Phi(e^{-i\epsilon}\theta), \quad A \rightarrow e^{i\epsilon R_\Phi} A, \quad (2.84)$$

$$\psi \rightarrow e^{i\epsilon(R_\Phi-1)}\psi, \quad R(\psi) = R(\Phi) - 1 \quad (2.85)$$

On the other hand, there is no room to rotate the vector superfield, since a nontrivial charge assignment for vector superfield contradicts the nonlinear coupling of vector multiplet in gauge interactions as given in eq.(2.57). The vector multiplet gives a relative phase rotation between boson and fermion as

$$V(\theta) \rightarrow V(e^{-i\epsilon}\theta), \quad (2.86)$$

$$\lambda_\alpha \rightarrow e^{i\epsilon}\lambda_\alpha, \quad R(\lambda) = +1 \quad (2.87)$$

We observe the following characteristic features in the R -symmetry.

1. R -symmetry is chiral. Therefore R -symmetry is generally anomalous.

If there is another anomalous chiral symmetry, usually a linear combination is anomaly free.

2. The mass term for the gaugino λ breaks the R -symmetry

$$\mathcal{L} = \frac{1}{2}m\lambda^\alpha\lambda_\alpha + h.c. \quad (2.88)$$

3. Superpotential P must have the R -charge $R(P) = 2$

$$\mathcal{L} = \frac{1}{2}D^2P(\Phi) \quad D^2 \approx \frac{\partial^2}{\partial\theta^2} \rightarrow e^{2i\epsilon}D^2, \quad (2.89)$$

Therefore possible terms in superpotential are restricted if one wishes to have the supersymmetric theory to be invariant under the R -symmetry transformation.

4. Phenomenologically it is desirable to break the R -symmetry explicitly. Since the massless gaugino is not observed in nature, R -symmetry should be broken as is seen from eq.(2.88). The explicit breaking of the R -symmetry will allow massive gauginos without encountering (light) R -axion resulting from the spontaneous breaking of the R -symmetry. To avoid a rapid proton decay, the R -parity $(-1)^R$ conservation is desirable replacing the continuous R -symmetry.

3 Supersymmetric $SU(3) \times SU(2) \times U(1)$ Model

3.1 Yukawa Coupling

3.1.1 Nonsupersymmetric Standard Model

Let us summarize the nonsupersymmetric $SU(2) \times U(1)$ model emphasizing the structure of the Yukawa couplings.

We have the (three) generations of the left-handed quark doublets q_j , the right-handed u -type quark singlets u_{Ri} , and the right-handed d -type quark singlets d_{Ri} . We also have the (three) generations of the left-handed lepton doublets l_j , and the right-handed electrons e_{Ri} . Here i, j, \dots indicates the generation indices.

We have complex Higgs doublets. Let us denote

$$\begin{pmatrix} \varphi_u \\ \varphi_d \end{pmatrix} \text{ Higgs to give masses to } \begin{pmatrix} u \\ d \end{pmatrix} \text{ type quark}$$

In terms of these fields, the Yukawa couplings f can be given by

$$L_{Yukawa} = f_u^{ij} \overline{u_{Ri}} \varphi_u^T \varepsilon q_j + f_d^{ij} \overline{d_{Ri}} \varphi_d^T \varepsilon q_j + f_e^{ij} \overline{e_{Ri}} \varphi_d^T \varepsilon l_j \quad (3.1)$$

where

$$q_i = \begin{pmatrix} u_i \\ d_i \end{pmatrix}, \quad l_i = \begin{pmatrix} \nu_i \\ e_i \end{pmatrix}, \quad (3.2)$$

$$\varphi_u = \begin{pmatrix} \varphi_u^+ \\ \varphi_u^0 \end{pmatrix}, \quad \varphi_d = \begin{pmatrix} \varphi_d^0 \\ \varphi_d^- \end{pmatrix} \quad (3.3)$$

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (3.4)$$

In the nonsupersymmetric model, nothing prevents choosing the Higgs doublet φ_u and φ_d to be the complex conjugate of each other

$$\varphi_u = \varepsilon \cdot \varphi_d^* \quad (3.5)$$

This is the choice in the nonsupersymmetric minimal standard model.

3.1.2 Supersymmetric Standard Model

It is important to note that the supersymmetric model requires the Yukawa interaction to be a term in the superpotential. This is an F -type term. The superfield in the Yukawa interaction should have the same chirality.

Therefore we need two Higgs doublet superfields H_u and H_d as separate negative chiral scalar superfields.

$$H_u \neq \varepsilon \cdot H_d^* \quad (3.6)$$

The supersymmetric Yukawa interaction is given by

$$L_{Yukawa} = -\theta P(\Phi)|_{\theta\theta} + h.c. \quad (3.7)$$

$$P = f_u^{ij} U_i^c H_u^T \varepsilon Q_j + f_d^{ij} D_i^c H_d^T \varepsilon Q_j + f_e^{ij} E_i^c H_d^T \varepsilon L_j + \mu H_u^T \varepsilon H_d \quad (3.8)$$

where we denoted the negative chiral scalar superfield by capital letters and the charge conjugate of the positive chiral scalar superfield in terms of the upper suffix c .

Higgsino (chiral fermion associated with the Higgs scalar) introduces anomaly in gauge currents. This anomaly has to be cancelled. Introducing the H_u and H_d as separate negative chiral scalar superfield serves to achieve the anomaly cancellation at the same time.

3.2 Particle Content

Now we find that we need at least a pair of Higgs doublet superfield, we will list the minimal particle content of the supersymmetric standard model. Our convention for the usual standard model $U(1)$ charge Y is

$$Q = I_3 + Y \tag{3.9}$$

The mixing occurs among the following fields

1. Chargino $\tilde{\varphi}_{u+}$ and \tilde{W}^+
2. Neutralino $\tilde{\varphi}_{u0}$, $\tilde{\varphi}_{d0}$, \tilde{W}^0 , \tilde{B}
3. Scalar left-right mixing \tilde{q} and \tilde{u}^c, \tilde{d}^c etc.

We obtain the following R -parity $(-1)^R$ to be conserved and there is no continuous R -symmetry.

- ordinary particles have $(-1)^R = +1$
- Supersymmetry particles which are denoted with $\tilde{}$, have $(-1)^R = -1$

	$J = 1$	$J = 1/2$	$J = 0$	I	Y	$SU(3)$
Gauge fields						
G	g_m	\tilde{g}				
W	W_m	\tilde{W}				
B	B_m	\tilde{B}				
Higgs field						
$H_u = \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix}$		$\tilde{\varphi}_u$	φ_u	$\frac{1}{2}$	$\frac{1}{2}$	
$H_d = \begin{pmatrix} H_d^0 \\ H_d^- \end{pmatrix}$		$\tilde{\varphi}_d$	φ_d	$\frac{1}{2}$	$-\frac{1}{2}$	
Quark field						
$Q_i = \begin{pmatrix} U_i \\ D_i \end{pmatrix}$		q_i	\tilde{q}_i	$\frac{1}{2}$	$\frac{1}{6}$	3
U_i^c		u_i^c	\tilde{u}_i^c	0	$-\frac{2}{3}$	3^*
D_i^c		d_i^c	\tilde{d}_i^c	0	$\frac{1}{3}$	3^*
Lepton field						
$L_i = \begin{pmatrix} N_i \\ E_i \end{pmatrix}$		l_i	\tilde{l}_i	$\frac{1}{2}$	$-\frac{1}{2}$	
E_i^c		e_i^c	\tilde{e}_i^c	0	1	
$(N_i^c$		ν_i^c	$\tilde{\nu}_i^c$	0	0)

We have denoted the possible right-handed neutrino superfield as N_i .

4 $N = 1$ Supersymmetry Nonperturbative Dynamics

4.1 Holomorphy

4.1.1 $N = 1$ Supersymmetry

The chiral scalar superfield contains the complex scalar field A as the first component as shown in eq.(2.32).

$$\Phi = A + \sqrt{2}\theta\psi + \theta^2 F \quad (4.1)$$

The distinction between negative chiral and positive chiral scalar superfield can be formulated as a distinction between holomorphic and anti-holomorphic fields. The former is associated with

the complex variable z , whereas the latter is associated with the complex conjugate variable \bar{z} .

Since there are terms restricted to the function of the chiral scalar superfield with only one of the chiralities, we obtain a restriction related to the distinction of holomorphic and anti-holomorphic quantities. The principle to distinguish the chirality is called the holomorphy and gives the following restrictions

1. The superpotential is restricted to be a holomorphic function.
2. The Kähler potential and the Fayet-Iliopoulos D -term are not controlled by holomorphy.

4.1.2 Complexified Symmetry Group

The principle of holomorphy gives the following consequences.

1. If a Lagrangian is invariant under a symmetry group G , it is automatically invariant under the complexification G^c of the symmetry group in the case of supersymmetric gauge theories, because of the holomorphy principle.
2. To maintain the supersymmetry, the auxiliary fields have to vanish.

$$F = 0 \tag{4.2}$$

This is a supersymmetric vacuum condition. One often finds parameters to specify the supersymmetric vacua. These parameters are called moduli.

It has been shown that the moduli in supersymmetric gauge theories are given by gauge invariant holomorphic functions constrained by $F = 0$ [25].

Because of holomorphy the manifold of vacuum states (moduli space) is invariant under complexified symmetry group G^c

3. It is usually most convenient to use the Wess-Zumino gauge to make the physical particle content manifest. The supersymmetric vacuum configuration in the Wess-Zumino gauge is given by the condition that both auxiliary fields should vanish: $F = 0$ and $D = 0$. Since the superpotential is invariant under the complexified symmetry group G^c , $F = 0$ condition is invariant under G^c . On the other hand, the kinetic term in the Wess-Zumino gauge is invariant under G , but not invariant under G^c . Therefore the condition $D = 0$ is not invariant under G^c .
4. For NonAbelian gauge group, or Abelian gauge group without the Fayet-Iliopoulos D -term, it is sufficient to impose the condition $F = 0$. Even if the condition $D = 0$ is not met by the field configuration in G^c , one can make a complexified gauge transformation to deform D to vanishing values $D = 0$. In this process, the condition $F = 0$ is unchanged because of the invariance of superpotential under the complexified gauge transformations.

4.1.3 Wilsonian action

In discussing the effective action for low energy field theories, we run across two different kind of the effective potentials.

1. Wilsonian effective action

$$Z = \int D\phi e^{-S_{bare}(\phi, \Lambda)} = \int D\phi_{<} e^{-S_{eff}(\phi_{<})} \quad (4.3)$$

$$e^{-S_{eff}} \equiv \int D\phi_{>} e^{-S_{bare}(\phi, \Lambda)} \quad (4.4)$$

We have denoted the modes with momenta larger than the scale μ as $\phi_{>}$, and the modes with momenta smaller than the scale μ as $\phi_{<}$.

In this definition, one integrates modes in momentum scales larger than the scale μ that one is interested in : $\phi_{>}$ in $\mu < p < \Lambda$. In this definition, one usually suppose that there is a cut-off in the momentum integration to make the integral meaningful and is denoted as Λ . Therefore this can be defined for nonrenormalizable theories as well. This definition has the advantage of receiving no infrared divergences. This feature avoids anomalies to holomorphy. Therefore the Wilsonian effective action S_{eff} is a holomorphic function of parameters and background fields. It is also noted that the beta function in the Wilsonian action is 1-loop exact in the $N = 1$ supersymmetric theories [27]. This can most easily be found that the trace anomaly is in the same supermultiplet as the axial anomaly, since the energy-momentum tensor, supercurrent, and the axial current are in the same supermultiplet :

$$(T^{mn}, S_{\alpha}^m, J^{5m}) \quad (4.5)$$

On the other hand, the axial anomaly is 1-loop exact according to the Adler-Bardeen theorem [26], whereas the trace anomaly gives the beta function. Therefore the trace anomaly is also one-loop exact provided one does not have anomaly in holomorphy.

2. One-Particle-Irreducible (1PI) effective action.

This is the usual effective action in the sense of the generating function for the one particle irreducible amputated amplitudes.

$$Z[J] = \int D\phi e^{-S(\phi) - J\phi} = e^{-W[J]} \quad (4.6)$$

$$\Phi \equiv \frac{\partial W}{\partial J} \quad (4.7)$$

$$\Gamma[\Phi] \equiv W[J] - J\Phi \quad (4.8)$$

If there are massless particles, this effective action usually has an infrared divergences which produces an anomaly for holomorphy. Therefore the beta function in the one particle irreducible effective action receives contributions from all orders of perturbation. More specifically, it can be computed from the knowledge of the one-loop beta function together with the anomalous dimension coming from the wave function renormalization.

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi} \frac{3T(G) - \sum_i T(R_i)(1 - \gamma_i)}{1 - \frac{T(G)\alpha}{2\pi}} \quad (4.9)$$

$$\gamma_i(\alpha) = -\frac{d \log Z(\mu)}{d \log \mu} = -C_2(R_i) \frac{\alpha}{\pi} + \dots \quad (4.10)$$

$$T^a T^a = C_2(R) \quad (4.11)$$

$$\text{tr}(T^a T^a) = T(R) \delta^{ab} \quad (4.12)$$

4.2 Nonperturbative Superpotential

The holomorphy and symmetry requirements restrict the superpotential severely in the case of $N = 1$ supersymmetric field theories. Quite often these requirements are enough to fix the superpotential P completely.

On the other hand, the Kähler potential is not holomorphic and is not constrained in the case of $N = 1$ supersymmetry. Therefore the kinetic term cannot be determined in the $N = 1$ supersymmetric theories. If we use the $N = 2$ supersymmetry, however, the kinetic term of the chiral scalar field associated with the vector multiplet is related to the kinetic term of the vector multiplet. Therefore there is a possibility to determine the Kähler potential nonperturbatively.

To find out the results on the nonperturbative effects, let us take the $SU(N_c)$ gauge group as an example. As for the matter multiplets, we take N_f flavors of "quark" and "antiquark" chiral scalar superfields Q and \tilde{Q} in the fundamental representation of $SU(N_c)$ gauge group.

$$Q_a^i, \quad \tilde{Q}_i^a \quad a = 1, \dots, N_c; \quad i = 1, \dots, N_f \quad (4.13)$$

4.2.1 $N_f < N_c$

Let us consider the massless supersymmetric QCD (SQCD) without superpotential.

$$\mathcal{L}_0 = \int d^4\theta \text{tr}\{Q^\dagger e^{2gV} Q + \tilde{Q} e^{-2gV} \tilde{Q}^\dagger\} \quad (4.14)$$

$$+ \frac{1}{2} \int d^2\theta \operatorname{tr} W^\alpha W_\alpha + \frac{1}{2} \int d^2\bar{\theta} \operatorname{tr} \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \quad (4.15)$$

The global symmetry in this theory at the classical level is given by

$$G_f = SU(N_f)_Q \times SU(N_f)_{\tilde{Q}} \times U(1)_B \times U(1)_A \times U(1)_X \quad (4.16)$$

Among them there are a number of Abelian global symmetries

$$Q(\theta) \rightarrow e^{i\alpha_B + i\alpha_A} Q(e^{-i\alpha_X} \theta) \quad (4.17)$$

$$\tilde{Q}(\theta) \rightarrow e^{-i\alpha_B + i\alpha_A} \tilde{Q}(e^{-i\alpha_X} \theta) \quad (4.18)$$

$$V(\theta) \rightarrow V(e^{-i\alpha_X} \theta) \quad (4.19)$$

The symmetry $U(1)_X$ is an R -type symmetry which make the relative rotation between bosons and fermions.

Let us illustrate how to determine the superpotential.

1. There is an anomaly in $U(1)_A$ and $U(1)_X$.

$$\partial_\mu j^\mu = \frac{1}{32\pi^2} \left[\sum_i q_i T(R_i) \right] F_{\mu\nu}^a \tilde{F}^{a\mu\nu} \quad (4.20)$$

$$\operatorname{tr}(t^a t^b) = \frac{1}{2} T(R) \delta^{ab} \quad (4.21)$$

We can define an anomaly free R -type symmetry $U(1)_R$ as a linear combination of $U(1)_A$ and $U(1)_X$. Then the anomaly free $U(1)$ quantum numbers are listed in the table.

Chiral Field	$U(1)_B$	$U(1)_R$
Q	1	$1 - N_c/N_f$
\tilde{Q}	-1	$1 - N_c/N_f$

2. Let us next find out the transformation property of the parameter which describes the strength of the gauge interaction Λ .

In order to see this, let us note that there is an instanton solution A_{inst}

$$F_{\mu\nu}^a(A_{\text{inst}}) = \tilde{F}_{\mu\nu}^a(A_{\text{inst}}) \equiv \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma a}(A_{\text{inst}}) \quad (4.22)$$

In this background, one finds that there are zero modes ψ^i_0 associated with the fermion field $\psi(x)$ whose number is determined by the index theorem.

$$\gamma^\mu D_\mu(A_{\text{inst}})\psi_0 = 0 \quad (4.23)$$

The number of zero modes for a chiral scalar field in the representation R is $T(R)$, which is the second Casimir for the representation. Similarly, the gauge fermion λ has $T(adj)$ of zero modes. The effective interaction among fermions can be found by considering the expectation value of an operator \mathcal{O}

$$\begin{aligned} \langle \mathcal{O} \rangle &= \int DAD\psi D\lambda e^{-S[A,\psi,\lambda]} \mathcal{O} \\ &\approx e^{-S[A_{\text{inst}}]} \int D\psi D\lambda e^{-\psi\gamma^\mu D_\mu(A_{\text{inst}})\psi - \lambda\gamma^\mu D_\mu(A_{\text{inst}})\lambda} \mathcal{O} \\ &\approx e^{-S[A_{\text{inst}}]} \int (D\psi D\lambda)_{\text{nonzero}} e^{-(\psi\gamma^\mu D_\mu(A_{\text{inst}})\psi)_{\text{nonzero}} - (\lambda\gamma^\mu D_\mu(A_{\text{inst}})\lambda)_{\text{nonzero}}} \\ &\quad \times \prod \int D\psi^i_0 D\lambda^i_0 \mathcal{O} \end{aligned} \quad (4.24)$$

where the value of the action at the instanton configuration is given by

$$S[A_{\text{inst}}] = -\frac{8\pi^2}{g^2} \quad (4.25)$$

Therefore we need to insert appropriate number of fermions in order to have nonvanishing contributions.

$$\left\langle \prod^{T(R)} \psi \prod^{T(adj)} \lambda \right\rangle \propto \exp\left(-\frac{8\pi^2}{g^2} + i\theta\right) = \Lambda^{3N_c - N_f} \quad (4.26)$$

where the coefficient of the one-loop beta function is given by $b = 3N_c - N_f$.

3. $U(1)_A$ transformation property of fermions are given for quarks and antiquarks by

$$\psi \rightarrow e^{i\alpha q} \psi, \quad q = 1 \quad (4.27)$$

for gauginos

$$\lambda \rightarrow e^{i\alpha q_\lambda} \lambda, \quad q_\lambda = 0 \quad (4.28)$$

Therefore the theory can be made invariant provided we assign the transformation property for the parameter Λ as

$$\langle \prod \psi \prod \lambda \rangle \rightarrow e^{i\alpha(2N_f q T(R) + q_\lambda T(adj))} \langle \prod \psi \prod \lambda \rangle \quad (4.29)$$

The above result shows that the theory itself is not invariant under this $U(1)_A$ transformation. Therefore it is anomalous. The amount of the anomaly is such that we can relate the (different) theory by assigning the above transformation property to the parameter of the theory, Λ . By this transformation, we are relating different theories. This property becomes useful when we determine the nonperturbative superpotential.

Therefore if we transform the parameter of the theory Λ as if it is a background field, we arrive at another theory related by the symmetry transformation. Hence there is a family of theories that are related by the transformations and the predictions of the theories are related by the transformation.

$$\Lambda^{3N_c-N_f} \rightarrow e^{i\alpha(2N_f qT(R)+q_\lambda T(adj))} \Lambda^{3N_c-N_f} \quad (4.30)$$

Namely $\Lambda^{3N_c-N_f}$ can be regarded as having $U(1)_A$ charge $2N_f qT(R) + q_\lambda T(adj) = 2N_f$.

One should imagine that the parameter to be a kind of background fields when one considers the transformation of the parameter of the theory. This method has been used extensively by Seiberg and collaborators [28].

4. Let us constrain the superpotential of the low energy effective action by demanding several requirements successively. The principle of holomorphy requires that the superpotential has to be a function of negative chiral scalar superfields only. Gauge invariance requires that the superpotential should be a function of gauge invariant combinations of superfields. Since $M_i^j = \tilde{Q}^a_i Q_a^j$ is the only color singlet negative chiral scalar superfield for the case $N_c > N_f$, we find that the superpotential should be a function of $M_i^j = \tilde{Q}^a_i Q_a^j$. Let us note that the holomorphy forbids to use the gauge invariant combination of negative and positive chiral scalar superfields such as $(Q_a^i)^* Q_a^j$. The global symmetry $SU(N_f) \times SU(N_f)$ dictates that the effective superpotential P should be a function of $\det(Q\tilde{Q})$ only.

$$P(Q, \tilde{Q}) = f(\det(Q\tilde{Q})) \quad (4.31)$$

Next we can use the transformation property under the (anomalous) global $U(1)_A$. As we have seen, the effective superpotential should be invariant under the transformation provided we assign a $U(1)_A$ charge for the parameter $\Lambda^{3N_c-N_f}$ as $2N_f qT(R) + q_\lambda T(adj) = 2N_f$. Therefore the superpotential should contain the parameter Λ as a function of the ratio $\Lambda^{3N_c-N_f} / \det(Q\tilde{Q})$ only. The dimensional analysis gives that the superpotential has to have the dimensions of M^3 . Thus superpotential is determined except overall numerical constants $C_{N_c N_f}$ that depend on N_c and N_f .

$$P = C_{N_c N_f} \left[\frac{\Lambda^{3N_c-N_f}}{\det(Q\tilde{Q})} \right]^{\frac{1}{N_c-N_f}}, \quad (4.32)$$

This set of numerical constants can be determined by two consistency conditions regarding the decoupling:

- (a) If we give a large mass to a quark Q_{N_f} , it should decouple. This relates the N_f case with $N_f - 1$ case with N_c unchanged.
- (b) If we give a large vacuum expectation value to a squark Q_i , the color gauge symmetry is partially broken and part of the flavor is decoupled. This relates the N_c, N_f case with $N_c - 1, N_f - 1$ case.

These two consistency conditions reduce the numerical coefficients to a single number C .

$$C_{N_c N_f} = (N_c - N_f) C^{\frac{1}{N_c - N_f}} \quad (4.33)$$

We can see that the Λ dependence of the $N_f = N_c - 1$ case agrees exactly with the one instanton contribution. Since the gauge symmetry is broken completely in this case, we can consider the large vacuum expectation values which corresponds to the weak coupling situation. Therefore we can trust the one-instanton calculation in this case and find

$$C = 1 \quad (4.34)$$

The resulting nonperturbative exact superpotential can be summarized as

$$P_{np} = \epsilon_{N_c - N_f} (N_c - N_f) \left[\frac{\Lambda^{3N_c - N_f}}{\det(Q\tilde{Q})} \right]^{\frac{1}{N_c - N_f}} \quad (4.35)$$

$$(\epsilon_{N_c - N_f})^{N_c - N_f} = 1 \quad (4.36)$$

If we consider the large vacuum expectation values for all the quark flavors, the gauge symmetry is broken from $SU(N_c)$ to $SU(N_c - N_f)$. The effective coupling between these two gauge theories should match at the scale of the vacuum expectation values. This matching condition reads

$$\left(\frac{\Lambda_{N_c, N_f}}{(\det \tilde{Q} Q)^{\frac{1}{2N_f}}} \right)^{3N_c - N_f} = \left(\frac{\Lambda_{N_c - N_f, 0}}{(\det \tilde{Q} Q)^{\frac{1}{2N_f}}} \right)^{3(N_c - N_f)} \quad (4.37)$$

For $N_f \leq N_c - 2$,

$$-\frac{8\pi^2}{g^2(\mu)} = \log \left(\frac{\Lambda}{\mu} \right)^b, \quad b = 3N_c - N_f \quad (4.38)$$

$$\begin{aligned}\mathcal{L} &= \frac{1}{4g^2} \int d^2\theta W^\alpha W_\alpha + \dots \\ &= -\frac{1}{32\pi^2} \log\left(\frac{\Lambda}{\mu}\right)^b \int d^2\theta W^\alpha W_\alpha + \dots\end{aligned}$$

The first component of the superpotential corresponds to the gaugino bilinear. Therefore the nonperturbative superpotential can be understood as gaugino condensation in the unbroken gauge group $SU(N_c - N_f)$

$$\frac{1}{32\pi^2} \langle 0 | \lambda^\alpha \lambda_\alpha | 0 \rangle = \epsilon_{N_c - N_f} \Lambda_{N_c - N_f, 0}^3 \quad (4.39)$$

So far we have discussed the nonperturbative effects in the $N = 1$ supersymmetric gauge theories. There has been much progress in recent years on the nonperturbative effects not only for the $N = 1$ supersymmetric theories but also for higher N supersymmetric theories that we have not enough space to cover. Among them it is worth mentioning that the exact solution for the low energy effective action of $N = 2$ supersymmetric gauge theories has been obtained up to two derivatives including the full nonperturbative effects [29].

5 Summary

1. Supersymmetry is the most promising solution to the gauge hierarchy problem.
2. Supersymmetry is the only nontrivial relativistic symmetry that relates particles with different spin.
3. Good progress has been made to understand the nonperturbative dynamics of supersymmetric gauge theories in both $N = 1$ and $N = 2$ supersymmetric theories .

Appendix A. Spinors and conventions

Our convention for the metric is given by $\eta_{mn} = (-1, +1, +1, +1)$ The γ matrices is defined in our convention by $(\gamma_m^{\text{here}} = \gamma_m^{\text{Wess-Bagger}} = \gamma_m^{\text{Bjorken-Drell}})$

$$\gamma_m \gamma_n + \gamma_n \gamma_m = -2\eta_{mn} \quad (1.1)$$

The conjugate spinor $\bar{\psi}$ for the spinor ψ is given by $\bar{\psi} \equiv \psi^\dagger \gamma_0 = -\psi^\dagger \gamma^0$. The chiral γ matrix γ_5 is defined by

$$\gamma_5 = \gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \gamma_5^{\text{Wess-Bagger}} = -i\gamma_5^{\text{Bjorken-Drell}} \quad (1.2)$$

It is useful to use the Weyl basis of γ matrix

$$\gamma_0 = -\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_j = \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \quad j = 1, 2, 3 \quad (1.3)$$

Combined together we introduce four dimensional notation for the two by two matrices $\sigma^m, \bar{\sigma}^m$

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix}, \quad \sigma^0 = \bar{\sigma}^0 \equiv 1, \quad \bar{\sigma}^j = -\sigma^j \quad (1.4)$$

In this basis, the chiral γ matrix becomes diagonal

$$\gamma_5 = -i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.5)$$

Since supersymmetry is conveniently formulated in terms of spinors of definite chirality, it is useful to decompose the usual four component spinor into upper and lower two component spinors with the definite chirality.

$$\psi \equiv \begin{pmatrix} \xi_\alpha \\ \eta^{*\dot{\alpha}} \end{pmatrix} \equiv \begin{pmatrix} \xi_\alpha \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix} \quad (1.6)$$

The negative and positive chirality spinors have undotted and dotted indices which are raised and/or lowered by antisymmetric ϵ tensor

$$\epsilon^{12} = -\epsilon_{12} = 1, \quad \epsilon_{\alpha\beta} \epsilon^{\beta\gamma} = \delta_\alpha^\gamma \quad (1.7)$$

The conjugate spinor is given by

$$\bar{\psi} = \left((\bar{\eta}^{\dot{\alpha}})^* \quad (\xi_\alpha)^* \right) = \left(\eta^\alpha \quad \xi^{*\dot{\alpha}} \right) \equiv \left(\eta^\alpha \quad \bar{\xi}_{\dot{\alpha}} \right) \quad (1.8)$$

$$\xi^\alpha \equiv \epsilon^{\alpha\beta} \xi_\beta, \quad \eta_{\dot{\alpha}} \equiv \epsilon_{\dot{\alpha}\dot{\beta}} \eta^{\dot{\beta}} \quad (1.9)$$

The charge conjugation matrix C is defined by

$$C^{-1} \gamma^m C = -\gamma^{mT} \quad (1.10)$$

One can show that C is antisymmetric and can be chosen to be unitary $C^T = -C$, $C^\dagger C = 1$. In the two-component notation using the Weyl basis, we have

$$C = -i\gamma_2\gamma_0 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \quad (1.11)$$

The charge conjugate spinor corresponds to antiparticle and is defined by

$$\psi^c \equiv C\bar{\psi}^T, \quad \bar{\psi}^c = -\psi^T C^{-1} \quad (1.12)$$

The charge conjugation reverses the chirality

$$\psi = \begin{pmatrix} \xi_\alpha \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix} \rightarrow \psi^c = \begin{pmatrix} \epsilon_{\alpha\beta}\eta^\beta \\ \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}_{\dot{\beta}} \end{pmatrix} = \begin{pmatrix} \eta_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix} \quad (1.13)$$

Spinors which are charge conjugate of itself is called the majorana spinor

$$\psi^c = \psi \rightarrow \psi = \begin{pmatrix} \eta_\alpha \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix} \quad \bar{\psi} = \begin{pmatrix} \eta^\alpha & \bar{\eta}_{\dot{\alpha}} \end{pmatrix} \quad (1.14)$$

Appendix B. Grassmann number and its derivatives

Grassmann number is defined as the anticommuting c-number. The derivative in terms of Grassmann number is defined by

$$\frac{\partial}{\partial\psi_\alpha}\psi_\beta = \delta_{\alpha\beta}, \quad \frac{\partial}{\partial\bar{\psi}_\alpha}\bar{\psi}_\beta = \delta_{\alpha\beta} \quad (1.1)$$

$$\frac{\partial}{\partial\psi_\alpha}\bar{\psi}_\beta = (C^{-1})_{\beta\alpha}, \quad \frac{\partial}{\partial\bar{\psi}_\alpha}\psi_\beta = (C)_{\beta\alpha} \quad (1.2)$$

$$\frac{\partial}{\partial\psi_\alpha} = \frac{\partial}{\partial\bar{\psi}_\beta}(C^{-1})_{\beta\alpha}, \quad \frac{\partial}{\partial\bar{\psi}_\alpha} = -(C)_{\alpha\beta}\frac{\partial}{\partial\psi_\beta} \quad (1.3)$$

$$\bar{\epsilon}\frac{\partial}{\partial\theta} = -\frac{\partial}{\partial\theta}\epsilon \quad (1.4)$$

Two-component notation

$$\frac{\partial}{\partial \eta_\alpha} \eta_\beta = \delta_\beta^\alpha, \quad \frac{\partial}{\partial \bar{\eta}^{\dot{\alpha}}} \bar{\eta}^{\dot{\beta}} = \delta_{\dot{\alpha}}^{\dot{\beta}} \quad (1.5)$$

$$\frac{\partial}{\partial \eta_\alpha} \eta^\beta = \epsilon^{\beta\alpha}, \quad \frac{\partial}{\partial \bar{\eta}^{\dot{\alpha}}} \bar{\eta}_{\dot{\beta}} = \epsilon_{\dot{\beta}\dot{\alpha}} \quad (1.6)$$

$$\frac{\partial}{\partial \eta^\alpha} \eta_\beta = \epsilon_{\beta\alpha}, \quad \frac{\partial}{\partial \bar{\eta}_{\dot{\alpha}}} \bar{\eta}^{\dot{\beta}} = \epsilon^{\dot{\beta}\dot{\alpha}} \quad (1.7)$$

$$\frac{\partial}{\partial \eta_\alpha} = \frac{\partial}{\partial \eta^\beta} \epsilon^{\beta\alpha}, \quad \frac{\partial}{\partial \bar{\eta}^{\dot{\alpha}}} = \frac{\partial}{\partial \bar{\eta}_{\dot{\beta}}} \epsilon_{\dot{\beta}\dot{\alpha}} \quad (1.8)$$

$$\frac{\partial}{\partial \eta^\alpha} = -\epsilon_{\alpha\beta} \frac{\partial}{\partial \eta_\beta}, \quad \frac{\partial}{\partial \bar{\eta}_{\dot{\alpha}}} = -\epsilon^{\dot{\alpha}\dot{\beta}} \frac{\partial}{\partial \bar{\eta}_{\dot{\beta}}} \quad (1.9)$$

$$\epsilon^\alpha \frac{\partial}{\partial \theta^\alpha} = -\frac{\partial}{\partial \theta_\alpha} \epsilon_\alpha, \quad \bar{\epsilon}_{\dot{\alpha}} \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} = -\frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} \bar{\epsilon}^{\dot{\alpha}} \quad (1.10)$$

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